The discovery of calculus is usually attributed to Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716), who worked independently in the late 1600s. Although Newton and Leibniz, along with their successors, discovered a number of properties of calculus, and calculus was found to have many applications in the physical sciences, it was not until the nineteenth century that a precise definition of a limit was proposed. Augustin Louis Cauchy (1789–1857), a French engineer and mathematician, gave this definition: "If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others." Even Cauchy, a master at rigor, was somewhat vague in his definition of a limit. What are "successive values," and what does it mean to "finally differ"? The phrase "finally differ from it by as little as one wishes" contains the seed of the ε–δ definition, because for the first time it indicates that the difference between $f(x)$ and its limit L can be made smaller than any given number, the number we labeled ε. The German mathematician Karl Weierstrass (1815–1897) first put together the definition that is equivalent to our ε–δ definition of a limit.

Concepts Review

1. The inequality $|f(x) - L| < \varepsilon$ is equivalent to ________ $< f(x) <$ ________.

2. The precise meaning of $\lim_{x \to a} f(x) = L$ is this: Given any positive number ε, there is a corresponding positive number δ such that ________ implies ________.

3. To be sure that $|3x - 3| < \varepsilon$, we would require that $|x - 1| <$ ________.

4. $\lim_{x \to a} (mx + b) =$ ________.

Problem Set 2.5

In Problems 1–6, give the appropriate ε–δ definition of each statement.

1. $\lim_{t \to a} f(t) = M$ **2.** $\lim_{u \to b} g(u) = L$

3. $\lim_{z \to d} h(z) = P$ **4.** $\lim_{y \to e} \phi(y) = B$

5. $\lim_{x \to c^-} f(x) = L$ **6.** $\lim_{t \to a^+} g(t) = D$

In Problems 7–18, give an ε–δ proof of each limit fact (see Examples 1–5).

7. $\lim_{x \to 0} (2x - 1) = -1$ **8.** $\lim_{x \to -21} (3x - 1) = -64$

9. $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5} = 10$ **10.** $\lim_{x \to 0} \left(\dfrac{2x^2 - x}{x}\right) = -1$

11. $\lim_{x \to 5} \dfrac{2x^2 - 11x + 5}{x - 5} = 9$ **12.** $\lim_{x \to 1} \sqrt{2x} = \sqrt{2}$

13. $\lim_{x \to 4} \dfrac{\sqrt{2x - 1}}{\sqrt{x - 3}} = \sqrt{7}$

14. $\lim_{x \to 1} \dfrac{14x^2 - 20x + 6}{x - 1} = 8$

15. $\lim_{x \to 1} \dfrac{10x^3 - 26x^2 + 22x - 6}{(x - 1)^2} = 4$

16. $\lim_{x \to 1} (2x^2 + 1) = 3$ **17.** $\lim_{x \to -1} (x^2 - 2x - 1) = 2$

18. $\lim_{x \to 0} x^4 = 0$

19. Prove that if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$ then $L = M$.

20. Let F and G be functions such that $0 \le F(x) \le G(x)$ for all x near c, except possibly at c. Prove that if $\lim_{x \to c} G(x) = 0$ then $\lim_{x \to c} F(x) = 0$.

21. Prove that $\lim_{x \to 0} x^4 \sin^2(1/x) = 0$. *Hint*: Use Problems 18 and 20.

22. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

23. By considering left- and right-hand limits, prove that $\lim_{x \to 0} |x| = 0$.

24. Prove that if $|f(x)| < B$ for $|x - a| < 1$ and $\lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} f(x)g(x) = 0$.

25. Suppose that $\lim_{x \to a} f(x) = L$ and that $f(a)$ exists (though it may be different from L). Prove that f is bounded on some interval containing a; that is, show that there is an interval (c, d) with $c < a < d$ and a constant M such that $|f(x)| \le M$ for all x in (c, d).

26. Prove that if $f(x) \le g(x)$ for all x in some deleted interval about a and if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then $L \le M$.