G. Continuity and Differentiability

**What you are finding:** Typical problems ask students to determine whether a function is continuous and/or differentiable at a point. Most functions that are given are continuous in their domain, and functions that are not continuous are not differentiable. So functions given usually tend to be piecewise and the question is whether the function is continuous and also differentiable at the $x$-value where the function changes from one piece to the other.

**How to find it:**

**Continuity:** I like to think of continuity as being able to draw the function without picking your pencil up from the paper. But to prove continuity at $x = c$, you have to show that $\lim_{x \to c} f(x) = f(c)$. Usually you will have to show that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

**Differentiability:** I like to think of differentiability as “smooth.” At the value $c$, where the piecewise function changes, the transition from one curve to another must be a smooth one. Sharp corners (like an absolute value curve) or cusp points mean the function is not differentiable there. The test for differentiability at $x = c$ is to show that $\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$. So if you are given a piecewise function, check first for continuity at $x = c$, and if it is continuous, take the derivative of each piece, and check that the derivative is continuous at $x = c$. Lines, polynomials, exponentials, sine and cosines curves are differentiable everywhere.

38. Let $f$ be the function defined below, where $c$ and $d$ are constants. If $f$ is differentiable at $x = -1$, what is the value of $c - d$?

$$f(x) = \begin{cases} x^2 + (2c + 1)x - d, & x \geq -1 \\ e^{2x+2} + cx + 3d, & x < -1 \end{cases}$$

A. -2   B. 0   C. 2   D. 3   E. 4

39. The graph of $f(x) = \sqrt{x^2 + 0.0001} - 0.01$ is shown in the graph to the right. Which of the following statements are true?

I. $\lim_{x \to 0} f(x) = 0$.

II. $f$ is continuous at $x = 0$.

III. $f$ is differentiable at $x = 0$.

A. I only   B. II only   C. I and II only   D. I, II, and III   E. None are true
40. Let \( f(x) \) be given by the function below. What values of \( a, b, \) and \( c \) do NOT make \( f(x) \) differentiable?

\[
f(x) = \begin{cases} 
  a \cos x, & x \leq 0 \\
  b \sin(x + c\pi), & x > 0 
\end{cases}
\]

A. \( a = 0, \ b = 0, \ c = 0 \)  
B. \( a = 0, \ b = 0, \ c = 100 \)  
C. \( a = 5, \ b = 5, \ c = 0.5 \)

D. \( a = -1, \ b = 1, \ c = 1 \)  
E. \( a = -8, \ b = 8, \ c = -2.5 \)

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41. Let \( f \) be the function defined below. Which of the following statements about \( f \) are NOT true?

\[
f(x) = \begin{cases} 
  x^3 - 1, & x < 1 \\
  \frac{x^3 - 1}{x-1}, & x \neq 1 \\
  3x, & x = 1 
\end{cases}
\]

I. \( f \) has a limit at \( x = 1 \).

II. \( f \) is continuous at \( x = 1 \).

III. \( f \) is differentiable at \( x = 1 \).

IV. The derivative of \( f' \) is continuous at \( x = 1 \).

A. IV only  
B. III and IV only  
C. II, III, and IV only  
D. I, II, III, and IV  
E. All statements are true